Applications of Chordal Flag Avergages

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Joint work with Tolga Birdal

University of Valencia
Image and Signal Processing Group

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Outline

Introduction

Averages over $\mathbb{R}$

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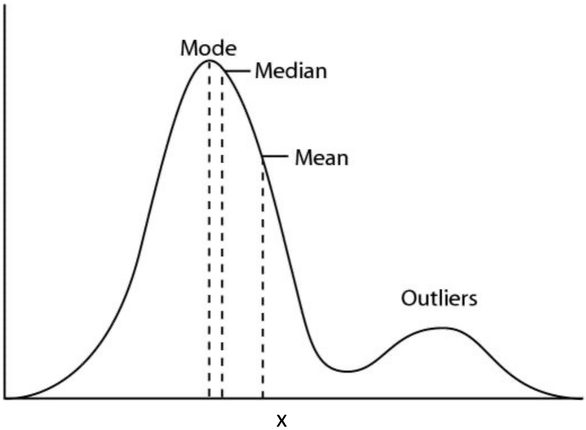

Figure: Hedges et al. 2003

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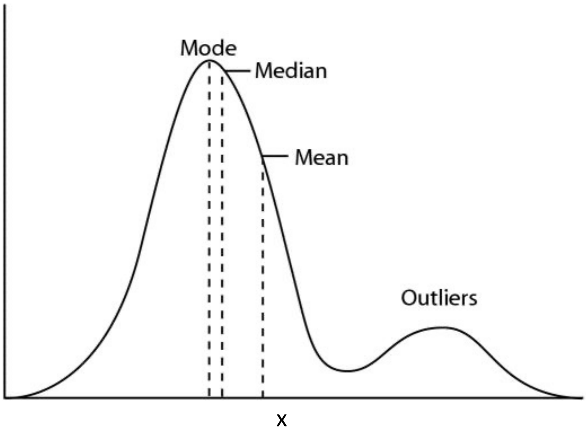

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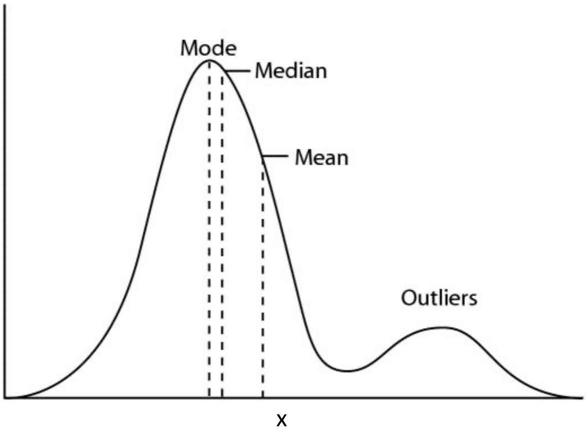

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The geometric median

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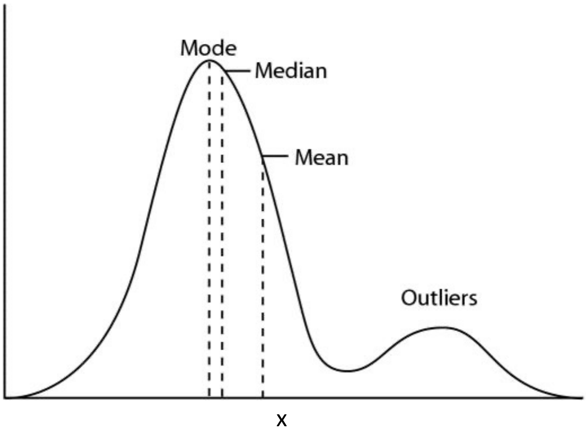

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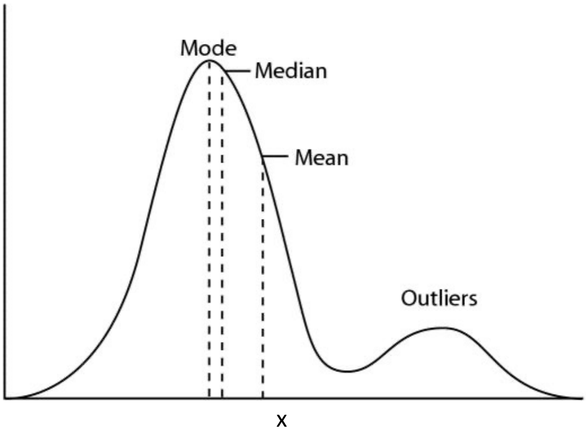

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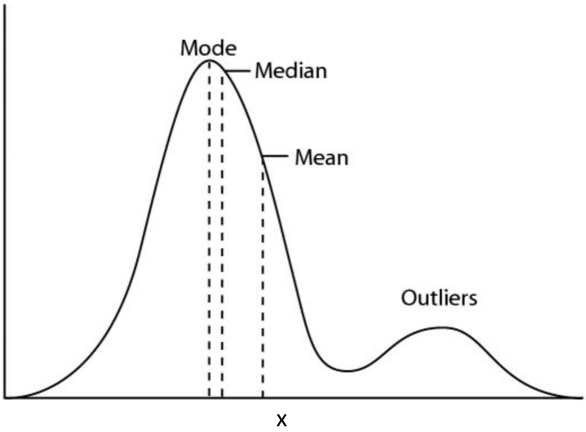

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Subspaces in Computer Vision

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This mean captures the common face across all three subspaces.

2D Embeddings of Flags vs. Subspaces

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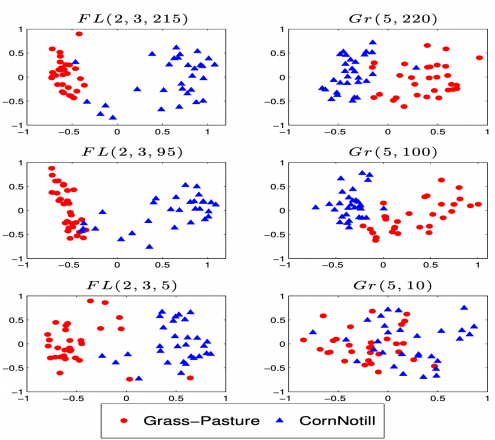

Figure: Indian Pines dataset. Ma et al. 2021

Flag representation (left) and subspace representation (right) for crops.

From Data to Flags

What is a Flag?

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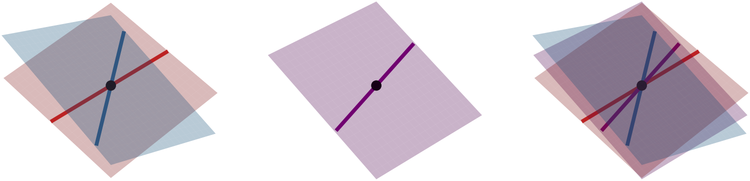

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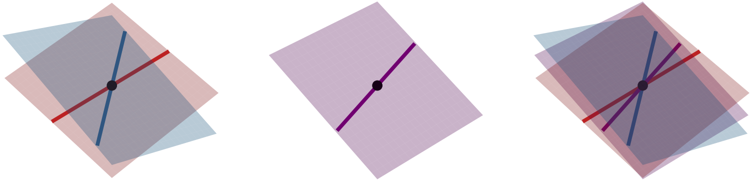

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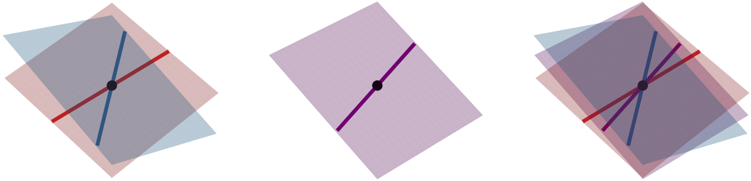

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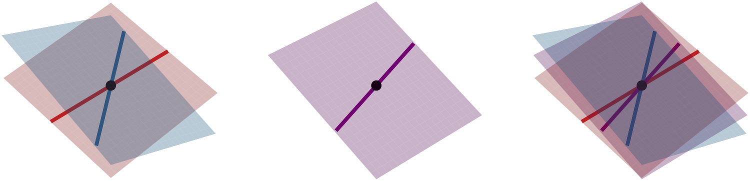

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Flag manifold of type (1,2,3)

$$\mathrm{FL}(1,2;3) = \left\{ [\![\mathbf{X}]\!] = \mathrm{span}(\mathbf{x}_1) \subset \mathrm{span}(\mathbf{x}_1, \mathbf{x}_2) \subset \mathbb{R}^3 \right\}$$

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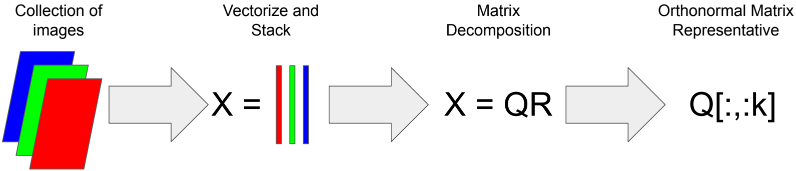

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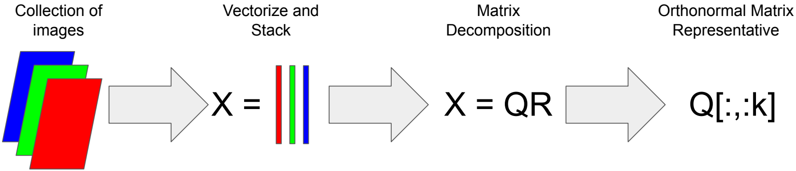

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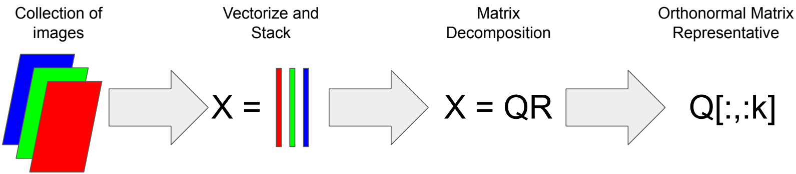

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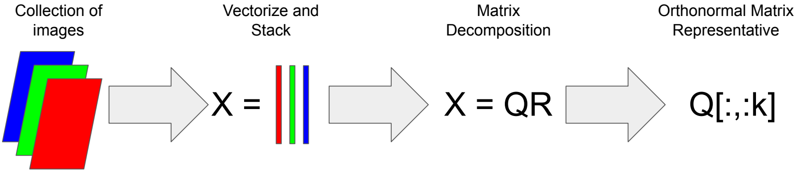

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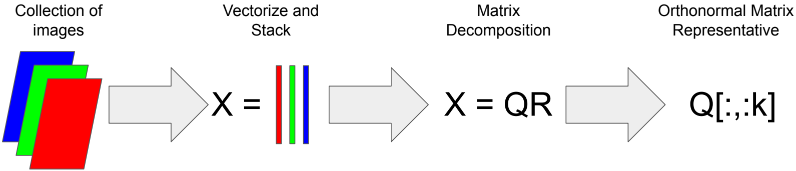

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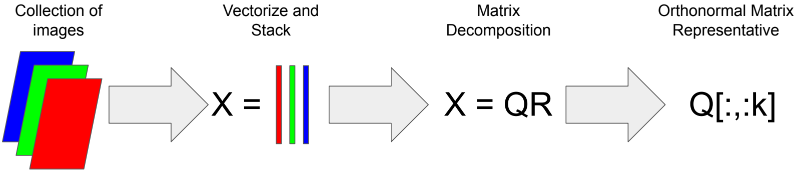

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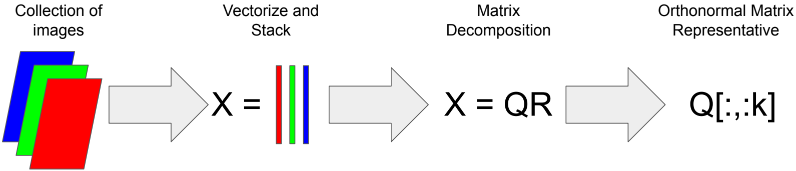

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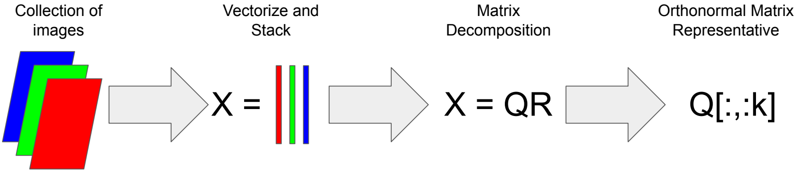

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Flags generalize subspaces by **maintaining the order** of the frames.

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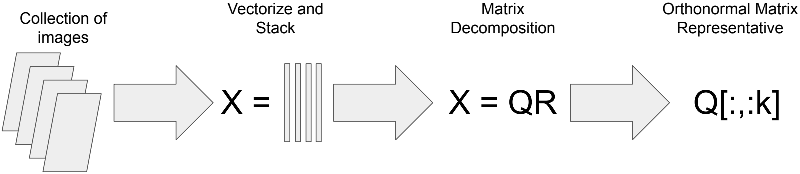

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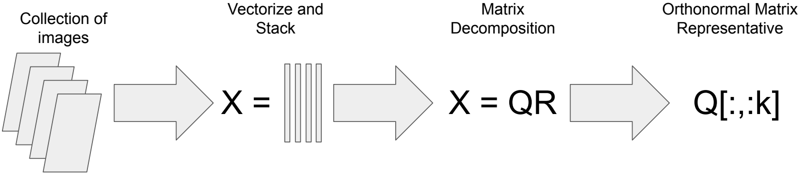

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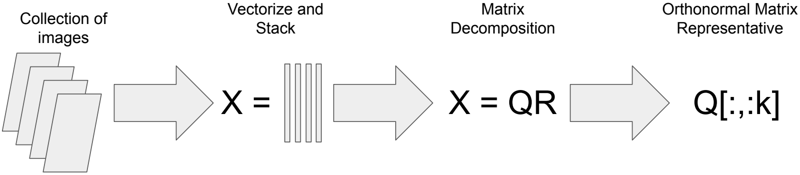

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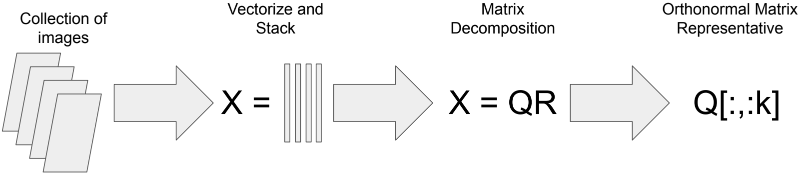

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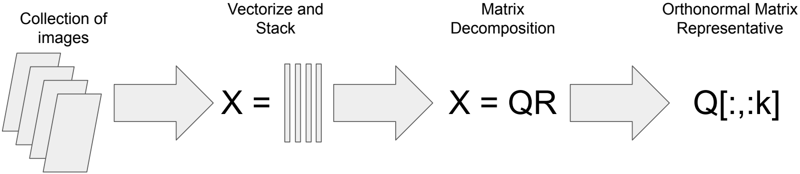

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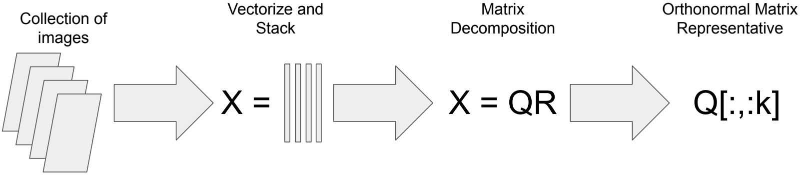

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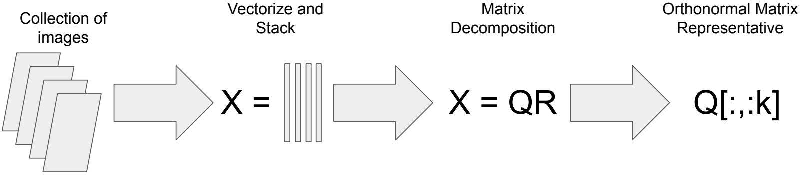

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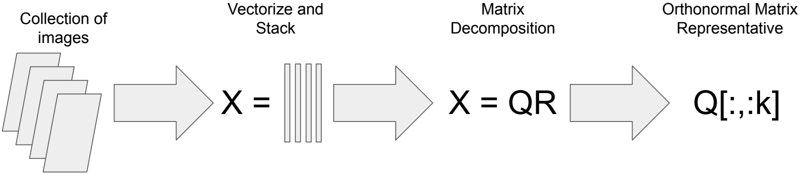

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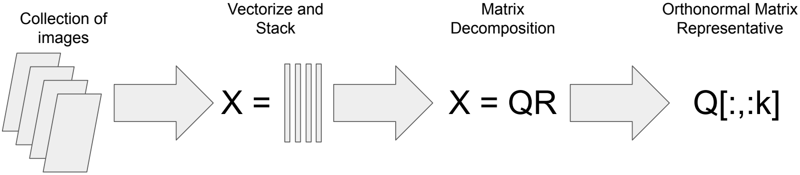

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Collect bands into groups of size l
$\implies$ Partial order of subspaces

$$\text{span}\{\mathbf{q}_1, ..., \mathbf{q}_l\} \subset \cdots \subset \text{span}\{\mathbf{q}_1, ..., \mathbf{q}_k\} \subset \mathbb{R}^n$$

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Weights: $\{\alpha_i\}_{i=1}^{p} \subset \mathbb{R}$

Chordal Flag Mean (Mankovich et al. 2023)

$$\underset{[\![\mathbf{Y}]\!] \in \mathbf{Fl}(n+1)}{\arg\min} \sum_{i=1}^{p} \alpha_i d_c([\![\mathbf{X}^{(i)}]\!], [\![\mathbf{Y}]\!])^2$$

Example:
Fl(k;n) solved. Flag Mean, (Draper et al. 2014)

Chordal Flag Median (Mankovich et al. 2023)

$$\underset{[\![\mathbf{Y}]\!] \in \mathbf{Fl}(n+1)}{\arg\min} \sum_{i=1}^{p} \alpha_i d_c([\![\mathbf{X}^{(i)}]\!], [\![\mathbf{Y}]\!])$$

Example:
Fl(k;n) FlagIRLS algorithm. Flag Median, (Mankovich et al. 2022)

Computing Chordal Flag Averages

The problem of averaging flags is largely unstudied.

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Chordal Flag Median: Algorithm Convergence (Mankovich et al. 2023)

An iteration of the chordal flag median algorithm does not* increase objective function values.

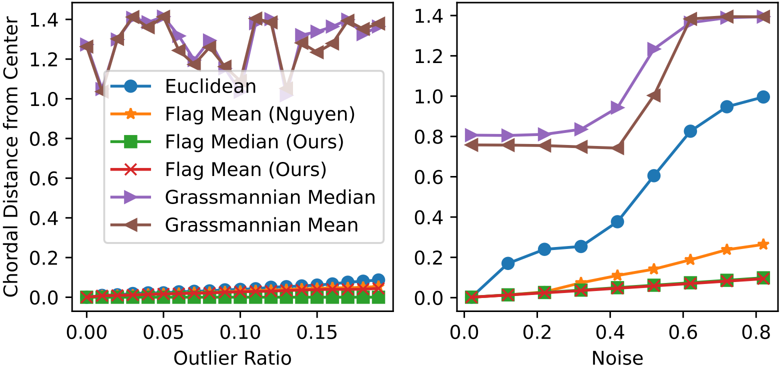
Examples

Flags vs. Subspaces

Flags vs. Subspaces

Flag averaging enforces the order of the faces.

Synthetic Data ($FL(1, 3; 10)$)



- Chordal flag averages- robust to outliers and noise
- Chordal flag mean- more robust estimate of chordal flag mean than Nguyen

Average motions represented as flags

$SE(3) = \{\text{motions in } \mathbb{R}^3\} = \{\text{rotation + translation in } \mathbb{R}^3\}$

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Averaging motions represented as flags is robust to noise and outliers compared to classical motion averaging methods.

MNIST Digits and NNs

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MNIST Digits = $\{$ images handwritten digits $\}$

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1. Dataset: 20 ones and *i* nines

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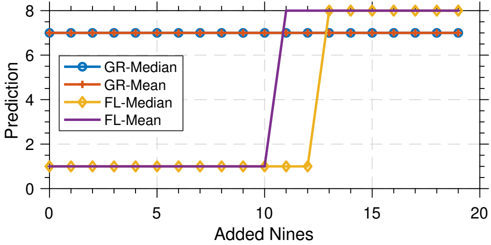
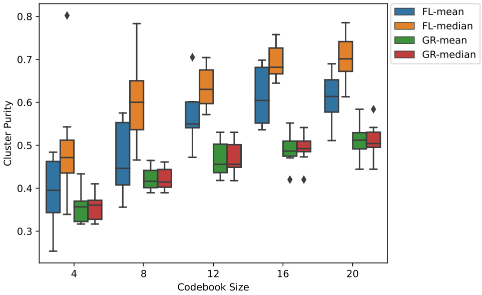


- Chordal flag averages are predicted correctly with $i = 1, 2, \dots, 10$ added nines.
- Median is more robust to outliers than mean.

LBG Clustering (UFC YouTube Dataset)

Linde-Buzo-Gray clustering using

- Distance: between flags (FL) vs. between subspaces (GR)
- Averages: between flags (FL) vs. between subspaces (GR)
- Averages: mean vs. median



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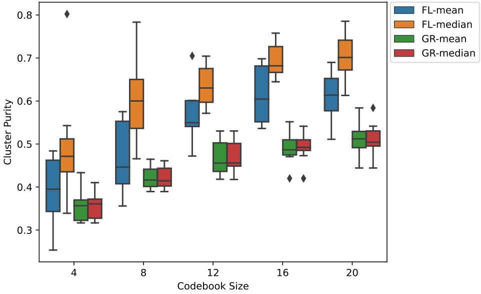

Flag averaging improves cluster purities over averages of subspaces.
Data from Liu et al. 2009.

Batch PCA

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1. Split samples into groups

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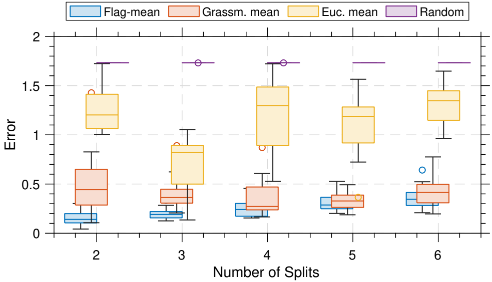

Chordal flag-means produce lower errors than the other methods.

Conclusion

Contributions

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For more information see *Chordal Flag Averaging and its Applications*.

Future Work

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Mathematical guarantees

- Convexity of chordal distance function
- Convergence rates for algorithms

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Mathematical guarantees

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Dimensionality reduction

- Finding PCA weights by optimizing over flags (Ye et al. 2023)
- Principal geodesic analysis, PCA on Riemannian manifolds (Fletcher et al. 2004)
- Robust versions of PCA (Tsaskiris et al. 2019 and Neumayer et al. 2019)
- Dynamic mode decomposition, kernels and more (Baddoo et al. 2021)
- Granger Causality and PCA (Varando et al. 2022)

Thank you.

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Backup Slides

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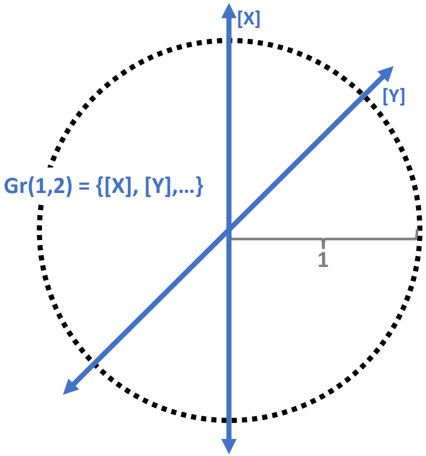
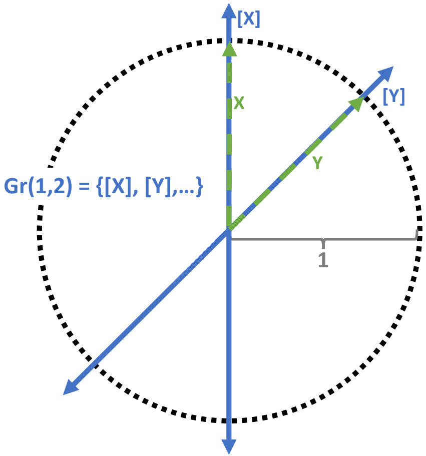
Distances Between Subspaces



Distances Between 1D Subspaces



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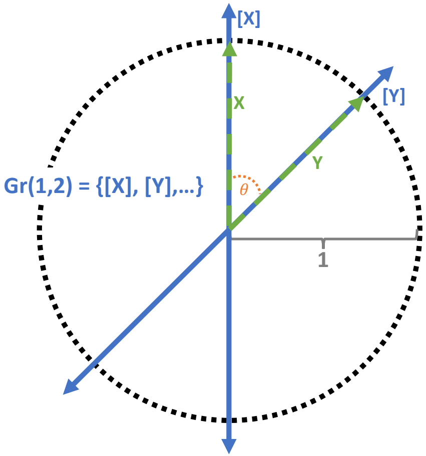

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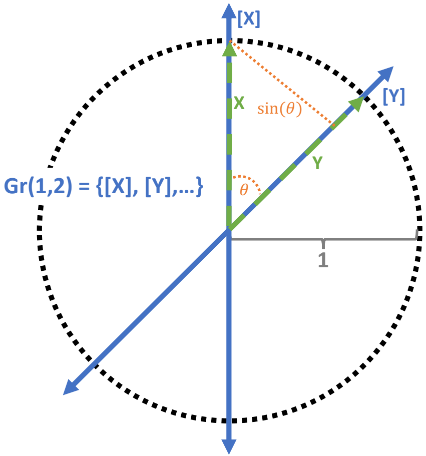

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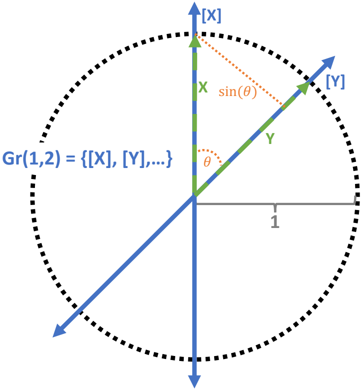


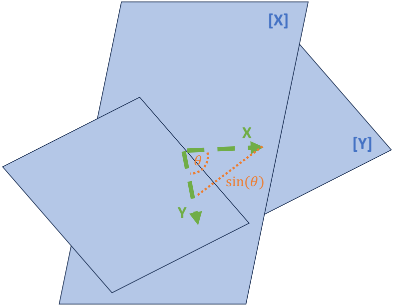
θ is the principal angle between [**X**] and [**Y**].

Geodesic distance is θ

Chordal distance is $\sin\theta$

Distances Between 2D Subspaces

Gr(2,3) = {[X], [Y],...}



There are k principal angles between subspaces of dimension k,

$$\theta([\mathbf{X}], [\mathbf{Y}]) = [\theta_1, \theta_2, \dots, \theta_k]^T.$$

Geodesic distance

$$\|\theta([\mathbf{X}], [\mathbf{Y}])\|_2$$

Chordal distance

$$\|\sin(\theta([\mathbf{X}], [\mathbf{Y}]))\|_2$$

Formalizing Subspace Prototypes

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Goal: Find central subspace prototype of $\{[\mathbf{X}_i]\}_{i=1}^{p} \in \mathrm{Gr}(k, n)$

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dist involves principal angles between subspaces (e.g., geodesic or chordal distance)!

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Average 20 examples of 7's and 8 examples of 6's from MNIST digits.

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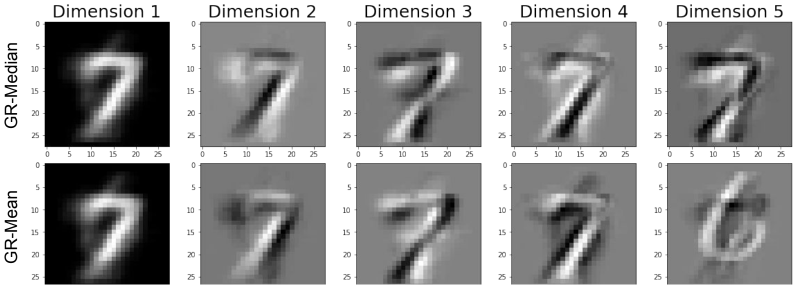

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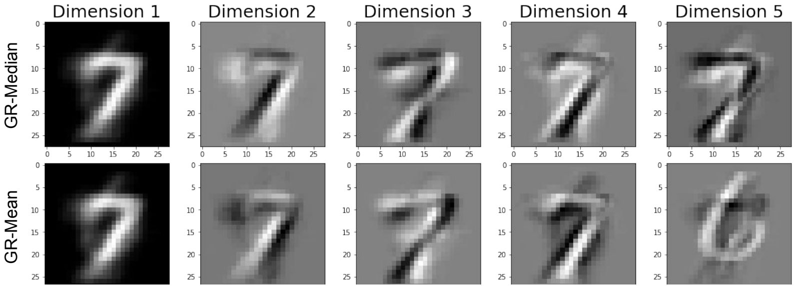

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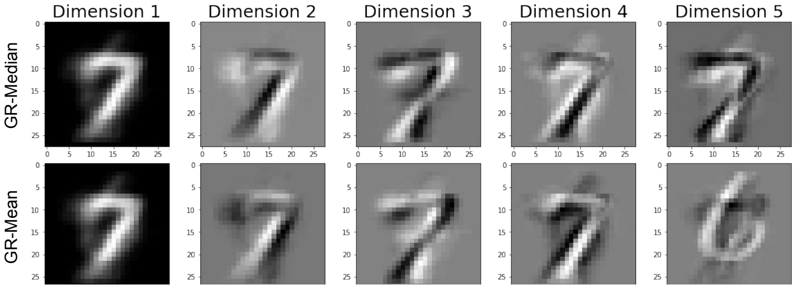

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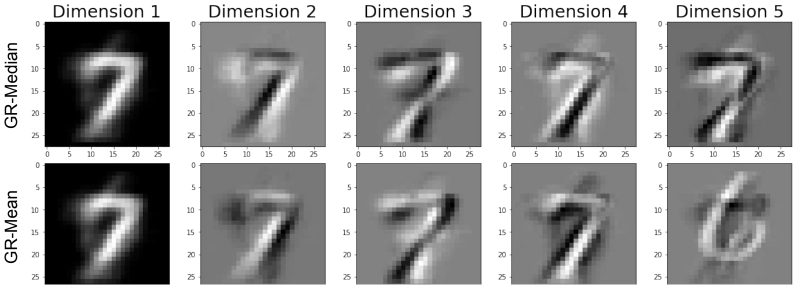

Chordal averages of subspaces are ordered subspaces.

Lower dimensions capture more signal than higher ones.

Median is more robust to outliers in dimension 5.

Algorithm for the Chordal Grassmannian Median

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Proposition (Mankovich 2023)

$f : \mathrm{Gr}(k,n) \to \mathbb{R}$ *chordal Grassmannian median objective function.*
$T : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ *iteration of algorithm.*

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$T : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ iteration of algorithm. Then

$$f(T([\mathbf{Y}])) \leq f([\mathbf{Y}]) + \frac{p\epsilon}{2}$$

More Examples of Flags

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Subspaces vs Flags of Subspaces

$$\text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \subset \mathbb{R}^4 \text{ vs. } \text{span}(\mathbf{x}_1) \subset \text{span}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \subset \mathbb{R}^4$$

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SVD

$\mathbf{X} \in \mathbb{R}^{n \times k}$, $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$

$$\text{span}(\mathbf{u}_1) \subset \text{span}(\mathbf{u}_1, \mathbf{u}_2) \subset \cdots \subset \text{span}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k) \subset \mathbb{R}^n$$

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Euclidean Motion (Selig 2005 and Ozyesil et al. 2018)

$$SE(3) \to SO(4) \to \text{FL}^+(1,2,3;4) \to SO(4) \to SE(3)$$

The Flag Manifold $\text{FL}(d_1, d_2, \dots, d_k; n)$

The Flag Manifold $\mathrm{FL}(d_1, d_2, \ldots, d_k; n)$

- Let S_i be a subspace. $\dim(S_i) = d_i$

$$\mathrm{FL}(n+1) = \mathrm{FL}(d_1, d_2, \ldots, d_k; n) = \{S_1 \subset S_2 \subset \cdots \subset S_k \subset \mathbb{R}^n\}$$

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See Ye et al. 2022 for more information

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Proposition (Mankoivich, Birdal 2023)

The chordal flag mean optimization problem is equivalent to

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The chordal flag median optimization problem can be approximated by a weighted chordal flag mean with weights

$$w_i(\mathbf{Y}) = \sum_{j=1}^{k} \frac{\alpha_i}{\max\left\{d_c([\![\mathbf{X}^{(i)}]\!], [\![\mathbf{Y}]\!]), \epsilon\right\}}.$$

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Let $[\![\mathbf{X}]\!], [\![\mathbf{Y}]\!] \in \mathrm{FL}(n+1)$.

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$$\begin{tikzcd}
{[[X]] = [X_1,} & {X_2,} & {\dots\ ,} & {X_k]} \\
{[[Y]] = [Y_1,} \arrow[u, leftrightarrow, "\mathrm{Gr}(m_1,n)"] & {Y_2,} \arrow[u, leftrightarrow, "\mathrm{Gr}(m_2,n)"] & {\dots\ ,} & {Y_k]} \arrow[u, leftrightarrow, "\mathrm{Gr}(m_k,n)"]
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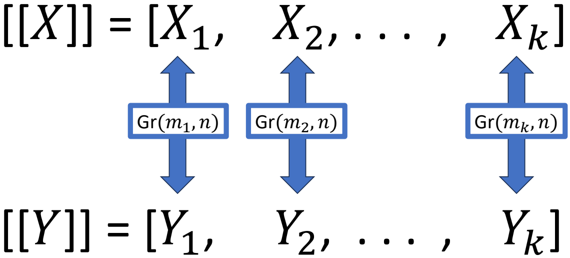

Chordal Distance Between Flags (Pitaval at al. 2013)

$$d_c([\![\mathbf{X}]\!], [\![\mathbf{Y}]\!]) := \sqrt{\sum_{j=1}^{k} \| \sin \theta([\mathbf{X}_j], [\mathbf{Y}_j]) \|_2^2}.$$

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